

Decay of the solution to the bipolar Euler-Poisson system with damping in \mathbb{R}^3

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Abstract: We construct the global solution to the Cauchy's problem of the bipolar Euler-Poisson equations with damping in \mathbb{R}^3 when H^3 norm of the initial data is small. If further, the \dot{H}^{-s} norm ($0 \leq s < 3/2$) or $\dot{B}_{2,\infty}^{-s}$ norm ($0 < s \leq 3/2$) of the initial data is bounded, we give the optimal decay rates of the solution. As a byproduct, the decay results of the $L^p - L^2$ ($1 \leq p \leq 2$) type hold without the smallness of the L^p norm of the initial data. In particular, we deduce that $\|\nabla^k(\rho_1 - \rho_2)\|_{L^2} \sim (1+t)^{-\frac{5}{4}-\frac{k}{2}}$ and $\|\nabla^k(\rho_i - \bar{\rho}, u_i, \nabla\phi)\|_{L^2} \sim (1+t)^{-\frac{3}{4}-\frac{k}{2}}$. We improve the decay results in Li and Yang [15] (*J. Differential Equations* 252(2012), 768-791), where they showed the decay rates as $\|\nabla^k(\rho_i - \bar{\rho})\|_{L^2} \sim (1+t)^{-\frac{3}{4}-\frac{k}{2}}$ and $\|\nabla^k(u_i, \nabla\phi)\|_{L^2} \sim (1+t)^{-\frac{1}{4}-\frac{k}{2}}$, when the $H^3 \cap L^1$ norm of the initial data is small. Our analysis is motivated by the technique developed recently in Guo and Wang [4] (*Comm. Partial Differential Equations* 37(2012), 2165-2208) with some modifications.

Key Words: Bipolar Euler-Poisson system; Global existence; Decay estimates; Negative Sobolev's space; Negative Besov's space.

1. Introduction

We consider the compressible bipolar Euler-Poisson equations with damping (BEP)

$$\begin{cases} \partial_t \rho_1 + \operatorname{div}(\rho_1 u_1) = 0, \\ \partial_t(\rho_1 u_1) + \operatorname{div}(\rho_1 u_1 \otimes u_1) + \nabla P(\rho_1) = \rho_1 \nabla \phi - \rho_1 u_1, \\ \partial_t \rho_2 + \operatorname{div}(\rho_2 u_2) = 0, \\ \partial_t(\rho_2 u_2) + \operatorname{div}(\rho_2 u_2 \otimes u_2) + \nabla P(\rho_2) = -\rho_2 \nabla \phi - \rho_2 u_2, \\ \Delta \phi = \rho_1 - \rho_2, \quad x \in \mathbb{R}^3, \quad t \geq 0, \end{cases} \quad (1.1)$$

where the unknown functions $\rho_i(x, t)$, $u_i(x, t)$ ($i = 1, 2$), $\phi(x, t)$ represent the charge densities, current densities, velocities and electrostatic potential, respectively, and the pressures $P = P(\rho_i)$ is a smooth function with $P'(\rho_i) > 0$ for $\rho_i > 0$. The system (1.1) is usually described charged particle fluids, for example, electrons and holes in semiconductor devices, positively and negatively charged ions in a plasma. We refer to [5, 19] for the physical background of the system (1.1).

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In this paper, we will study the global existence and large time behavior of the smooth solutions for the system (1.1) with the following initial data

$$\rho_i(x, 0) = \rho_{i0}(x) > 0, \quad u_i(x, 0) = u_{i0}(x), \quad i = 1, 2. \quad (1.2)$$

A lot of important works have been done on system (1.1). For one-dimensional case, we refer to Zhou and Li [31] and Tsuge [24] for the unique existence of the stationary solutions, Natalini [18] and Hsiao and Zhang [8] for global entropy weak solutions in the framework of compensated compactness on the whole real line and bounded domain respectively, Natalini [18], Hsiao and Zhang [9] for the relaxation-time limit, Gasser and Marcatti [3] for the combined limit, Huang and Li [7] for the large-time behavior and quasi-neutral limit of L^∞ -solution, Zhu and Hattori [32] for the stability of steady-state solutions to a recombined one-dimensional bipolar hydrodynamical model, Gasser, Hsiao and Li [2] for large-time behavior of smooth small solution.

For the multi-dimensional case, Lattanzio [10] discussed the relaxation limit, and Li [14] considered the diffusive relaxation. Ali and Jüngel [1] and Li and Zhang [13] studied the global smooth solutions of the Cauchy problem in the Sobolev's space and Besov's space, respectively. Later, Ju [6] investigated the global existence of smooth solution to the IBVP for the 3D bipolar Euler-Poisson system (1.1).

Recently, Using the classical energy method together with the analysis of the Green's function, Li and Yang [15] investigated the optimal decay rate of the Cauchy's problem of the system (1.1) of the classical solution when the initial data is small in the space $H^3 \cap L^1$. They deduced that the electric field (a nonlocal term in Hyperbolic-parabolic system) slows down the decay rate of the velocity of the BEP system, also see the recent works [11, 12, 30, 25, 27, 28] on the decay of the solutions to the unipolar Navier-Stokes-Poisson equations (NSP) and unipolar Euler-Poisson equations with damping. In fact, by the detailed analysis of the Green's function, it shows that the presence of the electric field slows down the decay rate in L^2 -norm of the velocity of the unipolar NSP system with the factor $1/2$ comparing with the Navier-Stokes system (NS) when the initial perturbation $\rho_0 - \bar{\rho}, u_0 \in L^p \cap H^3$ with $p \in [1, 2]$.

However, Wang [26] gave a different comprehension of the effect of the electric field on the time decay rates of the solution of the unipolar NSP system. The key idea is making an instead assumption on the initial perturbation $\rho_0 - \bar{\rho} \in \dot{H}^{-1}, u_0 \in L^2$. As a result, the electric field does not slow down but rather enhances the time decay rate of the density with the factor $\frac{1}{2}$. The method in [26] is initially established in Guo and Wang [4] for the estimates in the negative Sobolev's space. The proof in [4] is based on a family of energy estimates with minimum derivative counts and interpolations among them without linear decay analysis. Very recently, using this kind of energy estimates, Tan and Wang [22] discussed the Euler equations with damping in \mathbb{R}^3 , where they also gave the estimates in the negative Besov's space.

The main purpose of this paper is to improve the L^2 -norm decay estimates of the solutions in Li and Wang [15] by using this refined energy method together with the interpolation trick in [4, 26, 22]. Comparing with [4, 26, 22], the main additional difficulties are due to the presence of electronic field and the couple of two carriers by the Poisson equation. First, as Wang [26] pointed out, for the bipolar NSP system, there is one term $n_i u_i \nabla \phi$ can not be controlled by the dissipation terms when using this refined energy method, see the details in [26]. However, after an elaborate calculation, we can get each l -th ($l = 0, 1, 2, 3$) level energy estimate for the BEP system (1.1), see (2.26)-(2.28) and (2.38)-(2.39) in Lemma 2.10 and Lemma 2.11. Second, one can not obtain the dissipation term for $\|\rho_i\|_{L^2}$ in the energy estimates as the unipolar case in [26] since two species are strongly coupled by the Poisson equation for bipolar case. In fact, we only can get the estimate $\|\nabla^k(\rho_1 - \rho_2)\|_{L^2} \leq$

$\|\nabla^{k+1}\nabla\phi\|_{L^2}$ for the BEP system (1.1). As a result, one can not directly deal with the case $s \in (\frac{1}{2}, \frac{3}{2})$ for the estimates in the negative Sobolev's space or negative Besov's space by using the decay result for the case $s \in [0, \frac{1}{2}]$ as in [26]. In fact, the proof for the case $s \in (\frac{1}{2}, \frac{3}{2})$ in Wang [26] strongly depends on the derived decay result of the case $s = \frac{1}{2}$: $\|\rho\|_{L^2} \leq \|\nabla\nabla\phi\|_{L^2} \sim (1+t)^{-\frac{l+s}{2}} = (1+t)^{-\frac{3}{4}}$. After a detailed analysis, and by separating the cases that $s \in [0, \frac{1}{2}]$, $s \in (\frac{1}{2}, 1)$ and $s \in [1, \frac{3}{2})$ for the space \dot{H}^{-s} and $s \in [0, \frac{1}{2}]$, $s \in (\frac{1}{2}, 1)$, $s \in [1, \frac{3}{2})$ and $s = \frac{3}{2}$ for the space $\dot{B}_{2,\infty}^{-s}$, we achieve these estimates (See Lemma 2.13, Lemma 2.14 and Subsection 3.2).

Our main results are stated in the following theorems:

Theorem 1.1. Let $P'(\rho_i) > 0 (i = 1, 2)$ for $\rho_i > 0$, and $\bar{\rho} > 0$. Assume that $(\rho_i - \bar{\rho}, u_{i0}, \nabla\phi_0) \in H^3(\mathbb{R}^3)$ for $i = 1, 2$, with $\epsilon_0 =: \|(\rho_{i0} - \bar{\rho}, u_{i0}, \nabla\phi_0)\|_{H^3(\mathbb{R}^3)}$ small. Then there exists a unique, global, classical solution $(\rho_1, u_1, \rho_2, u_2, \phi)$ satisfying that for all $t \geq 0$,

$$\begin{aligned} & \|(n_1, u_1, n_2, u_2, \nabla\phi)\|_{H^3}^2 + \int_0^t \|(u_1, u_2)\|_{H^3}^2 + \|(\nabla n_1, \nabla n_2, \nabla(\nabla\phi))\|_{H^2}^2 d\tau \\ & \leq C\|(u_{10}, u_{20}, n_{10}, n_{20}, \nabla\phi_0)\|_{H^3}^2. \end{aligned} \quad (1.3)$$

Remark 1.1. From the fact $\nabla\phi_0 \in H^3$ is equivalent to $(n_{10} - n_{20}) \in H^2 \cap \dot{H}^{-1}$ deriving from the poisson equation (1.1)₅, we can replace the initial assumption $\nabla\phi_0 \in H^3$ by $n_{i0} \in H^2 \cap H^{-1}$.

Theorem 1.2. Under the assumptions of Theorem 1.1. If further, $(\rho_{i0} - \bar{\rho}, u_{i0}, \nabla\phi_0) \in \dot{H}^{-s}$ for some $s \in [0, 3/2)$ or $(\rho_{i0} - \bar{\rho}, u_{i0}, \nabla\phi_0) \in \dot{B}_{2,\infty}^{-s}$ for some $s \in (0, 3/2]$, then for all $t \geq 0$, there exists a positive constant C_0 such that

$$\|(\rho_i - \bar{\rho}, u_i, \nabla\phi)(t)\|_{\dot{H}^{-s}} \leq C_0 \quad (1.4)$$

or

$$\|(\rho_i - \bar{\rho}, u_i, \nabla\phi)(t)\|_{\dot{B}_{2,\infty}^{-s}} \leq C_0, \quad (1.5)$$

and

$$\|\nabla^l(\rho_i - \bar{\rho}, u_i, \nabla\phi)(t)\|_{H^{3-l}} \leq C_0(1+t)^{-\frac{l+s}{2}} \text{ for } l = 0, 1, 2; \ s \in [0, \frac{3}{2}); \quad (1.6)$$

$$\|\nabla^l(\rho_1 - \rho_2)(t)\|_{L^2} \leq C_0(1+t)^{-\frac{l+s+1}{2}} \text{ for } l = 0, 1; \ s \in [0, \frac{3}{2}]. \quad (1.7)$$

Remark 1.2. (1.7) is derived from (1.6) and the fact

$$\|\nabla^l(\rho_1 - \rho_2)\|_{L^2} = \|\nabla^l\Delta\phi\|_{L^2} \leq \|\nabla^{l+1}\nabla\phi\|_{L^2},$$

which shows the presence of the electric field enhances the time decay rate of disparity between two species.

Note that Lemma 2.4 (the Hardy-Littlewood-Sobolev theorem) implies that for $p \in (1, 2]$, $L^p \subset \dot{H}^{-s}$ with $s = 3(\frac{1}{p} - \frac{1}{2})$ and Lemma 2.6 implies that for $p \in [1, 2)$, $L^p \subset \dot{B}_{2,\infty}^{-s}$ with $s = 3(\frac{1}{p} - \frac{1}{2})$. Then Theorem 1.2 yields the following usual $L^p - L^2$ type of optimal decay results.

Corollary 1.1. Under the assumptions of Theorem 1.2 except that we replace the \dot{H}^{-s} or $\dot{B}_{2,\infty}^{-s}$ assumption by that $(\rho_{i0} - \bar{\rho}, u_{i0}, \nabla\phi_0) \in L^p$ for some $p \in [1, 2]$, then the following decay results hold:

$$\|\nabla^l(\rho_i - \bar{\rho}, u_i, \nabla\phi)(t)\|_{H^{3-l}} \leq C_0(1+t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{l}{2}}, \text{ for } l = 0, 1, 2; \quad (1.8)$$

$$\|\nabla^l(\rho_1 - \rho_2)(t)\|_{L^2} \leq C_0(1+t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{l+1}{2}}, \text{ for } l = 0, 1. \quad (1.9)$$

Remark 1.3. From Corollary 1.1, we know the each order derivatives of the density $\rho_i - \bar{\rho}$ and velocity u_i have the same decay rate in H^2 norm as the solutions of the Navier-Stokes equations. While the velocity u_i in [15] decays at the rate $(1+t)^{-\frac{1}{4}}$ in L^2 norm which is slower than the rate $(1+t)^{-\frac{3}{4}}$ for the compressible Navier-Stokes equations.

Remark 1.4. The energy method (close the energy estimates at each l -th level with respect to the spatial derivatives of the solutions) in this paper can not be applied to the bipolar Navier-Stokes-Poisson equations. In fact, as Wang [26] pointed out, there is one term $n_i u_i \nabla \phi$ can not be controlled by the dissipation terms, see the Introduction in [26]. Hence, it is also interesting to apply this energy method to the bipolar Navier-Stokes-Poisson equations.

Remark 1.5. We also notice that, the similar arguments can be used to investigate the full (nonisentropic) bipolar hydrodynamic models, which is under consideration.

Notations. In this paper, ∇^l with an integer $l \geq 0$ stands for the usual any spatial derivatives of order l . For $1 \leq p \leq \infty$ and an integer $m \geq 0$, we use L^p and $W^{m,p}$ denote the usual Lebesgue space $L^p(\mathbb{R}^n)$ and Sobolev spaces $W^{m,p}(\mathbb{R}^n)$ with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{m,p}}$, respectively, and set $H^m = W^{m,2}$ with norm $\|\cdot\|_{H^m}$ when $p = 2$. In addition, for $s \in \mathbb{R}$, we define a pseudo-differential operator Λ^s by

$$\Lambda^s g(x) = \int_{\mathbb{R}^n} |\xi|^s \hat{g}(\xi) e^{2\pi\sqrt{-1}x \cdot \xi} d\xi,$$

where \hat{g} denotes the Fourier transform of g . We define the homogeneous Sobolev's space \dot{H}^s of all g for which $\|g\|_{\dot{H}^s}$ is finite, where

$$\|g\|_{\dot{H}^s} := \|\Lambda^s g\|_{L^2} = \| |\xi|^s \hat{g} \|_{L^2}.$$

Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be such that $\eta(\xi) = 1$ when $|\xi| \leq 1$ and $\eta(\xi) = 0$ when $|\xi| \geq 2$. We define the homogeneous Besov's spaces $\dot{B}_{2,\infty}^{-s}(\mathbb{R}^3)$ with norm $\|\cdot\|_{\dot{B}_{2,\infty}^{-s}}$ defined by

$$\|f\|_{\dot{B}_{p,r}^{-s}} := \left(\sum_{j \in \mathbb{Z}} 2^{rsj} \|\dot{\Delta}_j f\|_{L^p}^r \right)^{\frac{1}{r}}.$$

Here $\dot{\Delta}_j f := F^{-1}(\varphi_j) * f$, $\varphi(\xi) = \eta(\xi) - \eta(2\xi)$ and $\varphi_j(\xi) = \varphi(2^{-j}\xi)$.

Throughout this paper, we will use a non-positive index s . For convenience, we will change the index to be “ $-s$ ” with $s \geq 0$. C or C_i denotes a positive generic (generally large) constant that may vary at different places. For simplicity, we write $\int f := \int_{\mathbb{R}^3} f dx$.

The rest of the paper is arranged as follows. In section 2, we give some useful Sobolev's inequality and Besov's inequality, then we give energy estimate in H^3 norm and some estimates in \dot{H}^{-s} and $\dot{B}_{2,\infty}^{-s}$. The proof of global existence and temporal decay results of the solutions will be derived in Section 3.

2. Nonlinear energy estimates

2.1. Preliminaries

In this subsection we give some Sobolev's inequalities and Besov's inequalities, which will be used in the next sections.

Lemma 2.1. (Gagliardo-Nirenberg's inequality). Let $0 \leq m, k \leq l$, then we have

$$\|\nabla^k g\|_{L^p} \leq C \|\nabla^m g\|_{L^q}^{1-\theta} \|\nabla^l g\|_{L^r}^\theta,$$

where k satisfies

$$\frac{1}{p} - \frac{k}{n} = (1-\theta) \left(\frac{1}{q} - \frac{m}{n} \right) + \theta \left(\frac{1}{r} - \frac{l}{n} \right).$$

Lemma 2.2. (Moser-type calculus) (i) Let $k \geq 1$ be an integer and define the commutator

$$[\nabla^k, g]h = \nabla^k(gh) - g\nabla^k h.$$

Then we have

$$\|[\nabla^k, g]h\|_{L^2} \leq C_k (\|\nabla g\|_{L^\infty} \|\nabla^{k-1} h\|_{L^2} + \|\nabla^k g\|_{L^2} \|h\|_{L^\infty}).$$

(ii) If $F(\cdot)$ is a smooth function, $f(x) \in H^k \cap L^\infty$, then we have

$$\|\nabla^k F(f)\| \leq C(k, F, \|f\|_{L^\infty}) \|\nabla^k f\|.$$

Lemma 2.3. ([4], Lemma A.5) Let $s \geq 0$ and $l \geq 0$, then we have

$$\|\nabla^l g\|_{L^2} \leq C \|\nabla^{l+1} g\|_{L^2}^{1-\theta} \|g\|_{H^{-s}}^\theta, \text{ where } \theta = \frac{1}{l+s+1}.$$

Lemma 2.4. ([21], Chapter V, Theorem 1) Let $0 < s < n, 1 < p < q < \infty, \frac{1}{q} + \frac{s}{n} = \frac{1}{p}$, then

$$\|\Lambda^{-s} g\|_{L^q} \leq C \|g\|_{L^p}.$$

Next, we give some lemmas on Besov space $\dot{B}_{2,\infty}^{-s}$.

Lemma 2.5. ([22]) Suppose $k \geq 0$ and $s > 0$, then we have

$$\|\nabla^k f\|_{L^2} \leq C \|\nabla^{k+1} f\|_{L^2}^{1-\theta} \|f\|_{\dot{B}_{2,\infty}^{-s}}^\theta, \text{ where } \theta = \frac{1}{l+1+s}.$$

Lemma 2.6. ([20]) Suppose that $s > 0$ and $1 \leq p < 2$. We have the embedding $L^p \subset \dot{B}_{q,\infty}^{-s}$ with $1/2 + s/3 = 1/p$. In particular we have the estimate

$$\|f\|_{\dot{B}_{2,\infty}^{-s}} \leq C \|f\|_{L^p}.$$

Lemma 2.7. ([20]) Suppose $k \geq 0$ and $s > 0$, then we have

$$\|\nabla^k\|_{L^2} \leq C \|\nabla^{k+1} f\|_{L^2}^{1-\theta} \|f\|_{\dot{B}_{2,\infty}^{-s}}^\theta, \text{ where } \theta = \frac{1}{l+s+1}.$$

Lemma 2.8. ([21]) If $1 \leq r_1 \leq r_2 \leq \infty$, then

$$\dot{B}_{2,r_1}^{-s} \in \dot{B}_{2,r_2}^{-s}.$$

Lemma 2.9. ([20]) If $m > l \geq k$ and $1 \leq p \leq q \leq r \leq \infty$. We have

$$\|g\|_{\dot{B}_{2,q}^l} \leq C \|g\|_{\dot{B}_{2,r}^k}^\theta \|g\|_{\dot{B}_{2p}^m}^{1-\theta},$$

where $l = k\theta + m(1-\theta)$, $\frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{p}$.

2.2 Energy estimates in H^3 -norm

We reformulate the nonlinear system (1.1) for $(\rho_1, u_1, \rho_2, u_2)$ around the equilibrium state $(\bar{\rho}, 0, \bar{\rho}, 0)$. Without loss of generality, we can assume $\bar{\rho} = 1$ and $P'(\bar{\rho}) = 1$. Denote

$$n_i = \rho_i - 1, \quad h(n_i) = \frac{P'(\rho_i)}{\rho_i} - 1,$$

then the Cauchy problem for $(n_1, u_1, n_2, u_2, \phi)$ is given by

$$\begin{cases} \partial_t n_1 + \operatorname{div} u_1 = -u_1 \cdot \nabla n_1 - n_1 \operatorname{div} u_1, \\ \partial_t u_1 + u_1 + \nabla n_1 - \nabla \phi = -u_1 \cdot \nabla u_1 - h(n_1) \nabla n_1, \\ \partial_t n_2 + \operatorname{div} u_2 = -u_2 \cdot \nabla n_2 - n_2 \operatorname{div} u_2, \\ \partial_t u_2 + u_2 + \nabla n_2 + \nabla \phi = -u_2 \cdot \nabla u_2 - h(n_2) \nabla n_2, \\ \Delta \phi = n_1 - n_2, \\ (n_1, u_1, n_2, u_2)(x, 0) = (\rho_{10} - 1, u_{10}, \rho_{20} - 1, u_{20})(x). \end{cases} \quad (2.1)$$

In this section, we will derive a priori nonlinear energy estimates for the equivalent system (2.1). Hence we assume a priori assumption that for a sufficiently small constant $\delta > 0$,

$$\|n_i(t)\|_{H^3} + \|u_i(t)\|_{H^3} + \|\nabla \phi(t)\|_{H^3} \leq \delta, \quad i = 1, 2, \quad (2.2)$$

which together with Sobolev's inequality, we have the facts

$$1/2 \leq n_i \leq 2, \quad |h^{(k)}(n_i)| \leq C, \quad i = 1, 2, \quad \text{for any } k \geq 0. \quad (2.3)$$

We first deduce the following energy estimates which contains the dissipation estimate for u_1, u_2 .

Lemma 2.10. Assume that $0 \leq k \leq 2$, then we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\nabla^k(n_1, u_1, n_2, u_2, \nabla \phi)|^2 + \|\nabla^k(u_1, u_2)\|_{L^2}^2 \\ & \leq C\delta(\|\nabla^{k+1}n_1\|_{L^2}^2 + \|\nabla^k u_1\|_{L^2}^2 + \|\nabla^{k+1}n_2\|_{L^2}^2 + \|\nabla^k u_2\|_{L^2}^2 + \|\nabla^{k+1}\nabla \phi\|_{L^2}^2). \end{aligned} \quad (2.4)$$

Proof. For $0 \leq k \leq 2$, applying ∇^k to (2.1)₁, (2.1)₂ and then multiplying the resulting equations by $\nabla^k n_1, \nabla^k u_1$ respectively, summing up and integrating over \mathbb{R}^3 , one has

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\nabla^k(n_1, u_1)|^2 + \|\nabla^k u_1\|_{L^2}^2 - \int \nabla^k u_1 \nabla^k \nabla \phi \\ & = - \int \nabla^k n_1 \nabla^k (u_1 \cdot \nabla n_1 + n_1 \operatorname{div} u_1) + \nabla^k u_1 \nabla^k (u_1 \cdot \nabla u_1 + h(n_1) \nabla n_1) \\ & = - \int \nabla^k (u_1 \cdot \nabla n_1) \nabla^k n_1 - \nabla^k (u_1 \cdot \nabla u_1) \nabla^k u_1 - \nabla^k (n_1 \operatorname{div} u_1) \nabla^k n_1 - \nabla^k (h(n_1) \nabla n_1) \nabla^k u_1 \\ & := I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (2.5)$$

We shall first estimate each term in the right hand side of (2.5). By Hölder's inequalities and Lemma 2.1, we get

$$\begin{aligned} I_1 & = - \sum_{0 \leq l \leq k} C_k^l \nabla^{k-l} u_1 \cdot \nabla \nabla^l n_1 \nabla^k n_1 \leq \sum_{0 \leq l \leq k} \|\nabla^{k-l} u_1 \nabla \nabla^l n_1\|_{L^{6/5}} \|\nabla^k n_1\|_{L^6} \\ & \leq \sum_{0 \leq l \leq k} \|\nabla^{k-l} u_1 \nabla \nabla^l n_1\|_{L^{6/5}} \|\nabla^{k+1} n_1\|_{L^2}. \end{aligned} \quad (2.6)$$

When $0 \leq l \leq [\frac{k}{2}]$, by Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} \|\nabla^{k-l} u_1 \nabla \nabla^l n_1\|_{L^{6/5}} &\leq \|\nabla^{k-l} u_1\|_{L^2} \|\nabla^{l+1} n_1\|_{L^3} \\ &\leq \|u_1\|_{L^2}^{\frac{l}{k}} \|\nabla^k u_1\|_{L^2}^{1-\frac{l}{k}} \|\nabla^\alpha n_1\|^{1-\frac{l}{k}} \|\nabla^{k+1} n_1\|_{L^2}^{\frac{l}{k}} \\ &\leq \delta(\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^k u_1\|_{L^2}), \end{aligned} \quad (2.7)$$

where α satisfies

$$l + \frac{3}{2} = \alpha(1 - \frac{l}{k}) + (k+1)\frac{l}{k},$$

which gives $\alpha = \frac{3k-2l}{2k-2l} \in [\frac{3}{2}, 3)$ since $l \leq \frac{k}{2}$.

When $[\frac{k}{2}] + 1 \leq l \leq k$, by Hölder's inequality and Lemma 2.1 again, we obtain

$$\begin{aligned} \|\nabla^{k-l} u_1 \nabla \nabla^l n_1\|_{L^{6/5}} &\leq \|\nabla^{k-l} u_1\|_{L^3} \|\nabla^{l+1} n_1\|_{L^2} \\ &\leq \|n_1\|_{L^2}^{\frac{k-l}{k+1}} \|\nabla^{k+1} n_1\|_{L^2}^{\frac{l+1}{k+1}} \|\nabla^\alpha u_1\|^{1-\frac{l+1}{k+1}} \|\nabla^{k+1} u_1\|_{L^2}^{\frac{l+1}{k+1}} \\ &\leq \delta(\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^k u_1\|_{L^2}), \end{aligned} \quad (2.8)$$

where α satisfies

$$k-l + \frac{1}{2} = \alpha \frac{l+1}{k+1} + k \frac{k-l}{k+1},$$

which implies $\alpha = \frac{3k-2l+1}{2l+2} \in [\frac{1}{2}, 3)$ since $l \geq \frac{k+1}{2}$.

From (2.6), (2.7) and (2.8), one has

$$I_1 \leq \delta(\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^k u_1\|_{L^2}). \quad (2.9)$$

For I_2 , using Lemma 2.1 and Hölder's inequality, we get

$$\begin{aligned} I_2 &= - \int ([\nabla^k, u_1] \nabla u_1 + u_1 \nabla \nabla^k u_1) \nabla^k u_1 \leq \|\nabla u_1\|_{L^\infty} \|\nabla^k u_1\|_{L^2}^2 - \frac{1}{2} \int u_1 \nabla (\nabla^k u_1 \nabla^k u_1) \\ &\leq \|\nabla u_1\|_{L^\infty} \|\nabla^k u_1\|_{L^2}^2 + \frac{1}{2} \int \operatorname{div} u_1 \nabla^k u_1 \cdot \nabla^k u_1 \leq \delta \|\nabla^k u_1\|_{L^2}^2. \end{aligned} \quad (2.10)$$

For I_3 ,

$$\begin{aligned} I_3 &= - \int \nabla^k (n_1 \operatorname{div} u_1) \nabla^k n_1 \\ &= - \int \sum_{0 \leq l \leq k-1} C_k^l \nabla^{k-l} n_1 \nabla^l \operatorname{div} u_1 \nabla^k n_1 - \int n_1 \operatorname{div} \nabla^k u_1 \nabla^k n_1 \\ &:= I_{31} + I_{32}. \end{aligned} \quad (2.11)$$

First, we estimate I_{31} . By Hölder's inequality, Lemma 2.1 and Cauchy's inequality, we obtain

$$\begin{aligned} I_{31} &= - \int \sum_{0 \leq l \leq k-1} C_k^l \nabla^{k-l} n_1 \nabla^l \operatorname{div} u_1 \nabla^k n_1 \\ &\leq C \sum_{0 \leq l \leq k-1} \|\nabla^{k-l} n_1 \nabla^l \operatorname{div} u_1\|_{L^{6/5}} \|\nabla^{k+1} n_1\|_{L^2}. \end{aligned} \quad (2.12)$$

When $0 \leq l \leq [\frac{k}{2}]$, using Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned} \|\nabla^{k-l} n_1 \nabla^l \operatorname{div} u_1\|_{L^{6/5}} &\leq C \|\nabla^{k-l} n_1\|_{L^2} \|\nabla^{l+1} u_1\|_{L^3} \\ &\leq C \|n_1\|_{L^2}^{\frac{l+1}{k+1}} \|\nabla^{k+1} n_1\|_{L^2}^{\frac{k-l}{k+1}} \|\nabla^\alpha u_1\|_{L^2}^{\frac{k-l}{k+1}} \|\nabla^k u_1\|_{L^2}^{\frac{l+1}{k+1}} \\ &\leq C \delta(\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^k u_1\|_{L^2}), \end{aligned} \quad (2.13)$$

where α satisfies

$$l + \frac{3}{2} = \alpha \frac{k-l}{k+1} + k \frac{l+1}{k+1},$$

which yields $\alpha = \frac{k+2l+3}{2k-2l} \in (\frac{1}{2}, 3)$ since $l \leq \frac{k}{2}$.

When $[\frac{k}{2}] + 1 \leq l \leq k-1$, using Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned} \|\nabla^{k-l} n_1 \nabla^l \operatorname{div} u_1\|_{L^{6/5}} &\leq C \|\nabla^{k-l} n_1\|_{L^3} \|\nabla^{l+1} u_1\|_{L^2} \\ &\leq C \|\nabla^\alpha n_1\|_{L^2}^{\frac{l+1}{k}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{k-1-l}{k}} \|u_1\|_{L^2}^{\frac{k-1-l}{k}} \|\nabla^k u_1\|_{L^2}^{\frac{l+1}{k}} \\ &\leq C \delta (\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^k u_1\|_{L^2}), \end{aligned} \quad (2.14)$$

where α satisfies

$$k-l + \frac{1}{2} = \alpha \frac{l+l}{k} + (k+1) \frac{k-l-1}{k},$$

which yields $\alpha = 1 + \frac{k}{2l+2} \in (\frac{3}{2}, 3)$ since $l \geq \frac{k+1}{2}$.

From (2.12), (2.13) and (2.14), we get

$$I_{31} \leq C \delta (\|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^k u_1\|_{L^2}^2). \quad (2.15)$$

For I_{32} , By Hölder's inequality, Lemma 2.1 and Cauchy's inequality, we obtain

$$\begin{aligned} I_{32} &= - \int n_1 \operatorname{div} \nabla^k u_1 \nabla^k n_1 \\ &= - \int n_1 \operatorname{div} (\nabla^k u_1 \nabla^k n_1) + \int n_1 \nabla^{k+1} n_1 \nabla^k u_1 \\ &\leq C \|\nabla n_1\|_{L^3} \|\nabla^k u_1\|_{L^2} \|\nabla^k n_1\|_{L^6} + \|n_1\|_{L^\infty} \|\nabla^{k+1} n_1\|_{L^2} \|\nabla^k u_1\|_{L^2} \\ &\leq C \delta (\|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^k u_1\|_{L^2}^2). \end{aligned} \quad (2.16)$$

Thus, (2.11), (2.15) and (2.16) imply

$$I_3 \leq C \delta (\|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^k u_1\|_{L^2}^2). \quad (2.17)$$

Next, we will estimate I_4 .

$$\begin{aligned} I_4 &= - \int \nabla^k (h(n_1) \nabla n_1) \nabla^k u_1 \\ &= - \int \sum_{0 \leq l \leq k} C_k^l \nabla^{k-l} h(n_1) \nabla^{l+1} n_1 \nabla^k u_1 + h(n_1) \nabla^{k+1} n_1 \cdot \nabla^k u_1 \\ &:= I_{41} + I_{42}. \end{aligned} \quad (2.18)$$

For I_{41} , By Hölder's inequality and Lemma 2.1, we obtain

$$\begin{aligned} I_{41} &= - \int \sum_{0 \leq l \leq k} C_k^l \nabla^{k-l} h(n_1) \nabla^{l+1} n_1 \nabla^k u_1 \\ &\leq C \|\nabla^{k-l} n_1 \nabla^{l+1} n_1\|_{L^2} \|\nabla^k u_1\|_{L^2}. \end{aligned} \quad (2.19)$$

When $0 \leq l \leq [\frac{k}{2}]$, by using Hölder's inequality and Lemma 2.1, we get

$$\begin{aligned} \|\nabla^{k-l} h(n_1) \nabla^{l+1} n_1\|_{L^2} &\leq \|\nabla^{k-l} h(n_1)\|_{L^6} \|\nabla^{l+1} n_1\|_{L^3} \\ &\leq C \|\nabla^{k-l} h(n_1)\|_{L^2}^{\frac{l}{k+1}} \|\nabla^{k+1} h(n_1)\|_{L^2}^{1-\frac{l}{k+1}} \|\nabla^\alpha n_1\|_{L^2}^{1-\frac{l}{k+1}} \|\nabla^{k+1} n_1\|_{L^2}^{\frac{l}{k+1}} \\ &\leq C \|\nabla^{k-l} n_1\|_{L^2}^{\frac{l}{k+1}} \|\nabla^{k+1} n_1\|_{L^2}^{1-\frac{l}{k+1}} \|\nabla^\alpha n_1\|_{L^2}^{1-\frac{l}{k+1}} \|\nabla^{k+1} n_1\|_{L^2}^{\frac{l}{k+1}} \\ &\leq C \delta \|\nabla^{k+1} n_1\|_{L^2}, \end{aligned} \quad (2.20)$$

where α satisfies

$$l + \frac{3}{2} = \alpha(1 - \frac{l}{k+1}) + l,$$

which implies $\alpha = \frac{3k+3}{2k-2l+2} \in [\frac{3}{2}, 3)$, since $l \leq \frac{k}{2}$.

When $[\frac{k}{2}] + 1 \leq l \leq k-1$, by Hölder's inequality and Lemma 2.1, we get

$$\begin{aligned} \|\nabla^{k-l} h(n_1) \nabla^{l+1} n_1\|_{L^2} &\leq \|\nabla^{k-l} h(n_1)\|_{L^3} \|\nabla^{l+1} n_1\|_{L^6} \\ &\leq C \|\nabla^\alpha h(n_1)\|_{L^2}^{\frac{l}{k-1}} \|\nabla^{k+1} h(n_1)\|_{L^2}^{1-\frac{l}{k-1}} \|\nabla^2 n_1\|_{L^2}^{1-\frac{l}{k-1}} \|\nabla^{k+1} n_1\|_{L^2}^{\frac{l}{k-1}} \\ &\leq C \|\nabla^\alpha n_1\|_{L^2}^{\frac{l}{k-1}} \|\nabla^{k+1} n_1\|_{L^2}^{1-\frac{l}{k-1}} \|\nabla^2 n_1\|_{L^2}^{1-\frac{l}{k-1}} \|\nabla^{k+1} n_1\|_{L^2}^{\frac{l}{k-1}} \\ &\leq C\delta \|\nabla^{k+1} n_1\|_{L^2}, \end{aligned} \tag{2.21}$$

where α satisfies

$$k-l + \frac{1}{2} = \alpha \frac{l}{k-1} + (k+1)(1 - \frac{l}{k-1}),$$

which implies $\alpha = 2 + \frac{-k+1}{2l} \in [\frac{3}{2}, 3)$ since $l \geq \frac{k+1}{2}$.

Thus, from (2.18), (2.19), (2.20) and (2.21), we deduce that

$$I_4 \leq C\delta (\|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^k u_1\|_{L^2}^2). \tag{2.22}$$

Hence, for n_1 and u_1 , we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int |\nabla^k(n_1, u_1)|^2 + \|\nabla^k u_1\|_{L^2}^2 - \int \nabla^k u_1 \nabla^k \nabla \phi \\ &\leq C\delta (\|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^k u_1\|_{L^2}^2). \end{aligned} \tag{2.23}$$

In the same way, we can get the following estimates for n_2 and u_2 , that is,

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int |\nabla^k(n_2, u_2)|^2 + \|\nabla^k u_2\|_{L^2}^2 + \int \nabla^k u_2 \nabla^k \nabla \phi \\ &\leq C\delta (\|\nabla^{k+1} n_2\|_{L^2}^2 + \|\nabla^k u_2\|_{L^2}^2). \end{aligned} \tag{2.24}$$

Finally, we will turn to estimate the last term in left hand side of (2.23) and (2.24). Since n_1 and n_2 is coupled in Poisson equation, we will estimate them simultaneously as follows.

$$\begin{aligned} &-\int \nabla^k \nabla \phi \cdot \nabla^k u_1 + \int \nabla^k \nabla \phi \cdot \nabla^k u_2 \\ &= \int \nabla^k (\operatorname{div} u_1) \nabla^k \phi - \int \nabla^k (\operatorname{div} u_2) \nabla^k \phi \\ &= -\int \nabla^k [\partial_t n_1 + \operatorname{div}(n_1 u_1)] \nabla^k \phi + \int \nabla^k [\partial_t n_2 + \operatorname{div}(n_2 u_2)] \nabla^k \phi \\ &= -\int \nabla^k \partial_t (n_1 - n_2) \nabla^k \phi - \int \nabla^k (\operatorname{div}(n_1 u_1)) \nabla^k \phi + \int \nabla^k (\operatorname{div}(n_2 u_2)) \nabla^k \phi \\ &= -\int \nabla^k \partial_t \Delta \phi \nabla^k \phi - \int \nabla^k (\operatorname{div}(n_1 u_1)) \nabla^k \phi + \int \nabla^k (\operatorname{div}(n_2 u_2)) \nabla^k \phi \\ &= \frac{1}{2} \frac{d}{dt} \|\nabla^k \nabla \phi\|_{L^2}^2 + \int \nabla^k (n_1 u_1) \nabla^k \nabla \phi - \int \nabla^k (n_2 u_2) \nabla^k \nabla \phi \\ &:= \frac{1}{2} \frac{d}{dt} \|\nabla^k \nabla \phi\|_{L^2}^2 + I_{51} + I_{52} \end{aligned} \tag{2.25}$$

Now we will estimate I_{51} and I_{52} .

When $k = 0$, by Hölder's inequality, Sobolev's inequality and Cauchy's inequality, we have

$$\begin{aligned} \int n_1 u_1 \nabla \phi &\leq C \|\nabla \phi\|_{L^6} \|u_1\|_{L^2} \|n_1\|_{L^3} \leq C \|\nabla \nabla \phi\|_{L^2} \|u_1\|_{L^2} \|n_1\|_{L^3} \\ &\leq C \delta (\|\nabla \nabla \phi\|_{L^2}^2 + \|u\|_{L^2}^2). \end{aligned} \quad (2.26)$$

Similarly, for $k = 1$, we get

$$\begin{aligned} \int \nabla(n_1 u_1) \nabla(\nabla \phi) &= - \int (n_1 u_1) \nabla^2 \nabla \phi \leq C \|\nabla^2 \nabla \phi\|_{L^2} \|u\|_{L^6} \|n\|_{L^3} \\ &\leq C \|\nabla^2 \nabla \phi\|_{L^2} \|\nabla u_1\|_{L^2} \|n\|_{L^3} \\ &\leq C \delta (\|\nabla^2 \nabla \phi\|_{L^2}^2 + \|\nabla u_1\|_{L^2}^2); \end{aligned} \quad (2.27)$$

and for $k = 2$, we have

$$\begin{aligned} \int \nabla^2(n_1 u_1) \nabla^2(\nabla \phi) &= - \int \nabla(n_1 u_1) \nabla^3 \nabla \phi \leq \|\nabla^3 \nabla \phi\|_{L^2} \sum_{0 \leq l \leq 1} \|\nabla^{1-l} n_1 \nabla^l u_1\|_{L^2} \\ &\leq C \|\nabla^3 \nabla \phi\|_{L^2} \|\nabla^\alpha n_1\|_{L^2}^{\frac{l+1}{2}} \|\nabla^3 n_1\|_{L^2}^{1-\frac{l+1}{2}} \|u_1\|_{L^2}^{1-\frac{l+1}{2}} \|\nabla^2 u_1\|_{L^2}^{\frac{l+1}{2}} \\ &\leq C \delta (\|\nabla^3 \nabla \phi\|_{L^2}^2 + \|\nabla^3 n_1\|_{L^2}^2 + \|\nabla^2 u_1\|_{L^2}^2), \end{aligned} \quad (2.28)$$

where

$$\alpha = \frac{l}{l+1}, \quad l = 0, 1.$$

In the same way, one can obtain the estimate of I_{52} . Hence, from (2.25) to (2.28), we have

$$I_{51} + I_{52} \geq -C \delta (\|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^k u_1\|_{L^2}^2 + \|\nabla^{k+1} n_2\|_{L^2}^2 + \|\nabla^k u_2\|_{L^2}^2 + \|\nabla^{k+1} \nabla \phi\|_{L^2}^2). \quad (2.29)$$

Combining (2.23), (2.24), (2.25) and (2.29), we deduce that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int |\nabla^k(n_1, u_1, n_2, u_2, \nabla \phi)|^2 + \|\nabla^k(u_1, u_2)\|_{L^2}^2 \\ &\leq C \delta (\|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^k u_1\|_{L^2}^2 + \|\nabla^{k+1} n_2\|_{L^2}^2 + \|\nabla^k u_2\|_{L^2}^2 + \|\nabla^{k+1} \nabla \phi\|_{L^2}^2). \end{aligned} \quad (2.30)$$

This proves Lemma 2.10. \blacksquare

Next, we derive the second type of energy estimates excluding n_1, u_1 and n_2, u_2 themselves.

Lemma 2.11. Assume that $0 \leq k \leq 2$, then we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int |\nabla^{k+1}(n_1, u_1, n_2, u_2, \nabla \phi)|^2 + \|\nabla^{k+1}(u_1, u_2)\|_{L^2}^2 \\ &\leq C \delta (\|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2 + \|\nabla^{k+1} n_2\|_{L^2}^2 + \|\nabla^{k+1} u_2\|_{L^2}^2 + \|\nabla^{k+1} \nabla \phi\|_{L^2}^2). \end{aligned} \quad (2.31)$$

Proof. For $0 \leq k \leq 2$, applying ∇^{k+1} to (2.1)₁, (2.1)₂ and then multiplying the resulting equations by $\nabla^{k+1} n_1, \nabla^{k+1} u_1$ respectively, summing up and integrating over \mathbb{R}^3 , one has

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int |\nabla^{k+1}(n_1, u_1)|^2 + \|\nabla^{k+1} u_1\|_{L^2}^2 - \int \nabla^{k+1} u_1 \cdot \nabla^{k+1} \nabla \phi \\ &= - \int \nabla^{k+1} n_1 \nabla^{k+1} (u_1 \cdot \nabla n_1 + n_1 \operatorname{div} u_1) + \nabla^{k+1} u_1 \nabla^{k+1} (u_1 \cdot \nabla u_1 + h(n_1) \nabla n_1) \\ &= - \int [\nabla^{k+1} (u_1 \cdot \nabla n_1) \nabla^{k+1} n_1 + \nabla^{k+1} (u_1 \cdot \nabla u_1) \nabla^{k+1} u_1] \\ &\quad - \int [\nabla^{k+1} (n_1 \operatorname{div} u_1) \nabla^{k+1} n_1 + \nabla^{k+1} (h(n_1) \nabla n_1) \nabla^{k+1} u_1] \\ &:= J_1 + J_2. \end{aligned} \quad (2.32)$$

Now we shall estimate J_1 and J_2 . By Lemma 2.2, Hölder's inequality and Cauchy's inequality, we get

$$\begin{aligned}
J_1 &= - \int \nabla^{k+1}(u_1 \cdot \nabla n_1) \nabla^{k+1} n_1 + \nabla^{k+1}(u_1 \cdot \nabla u_1) \nabla^{k+1} u_1 \\
&= - \int [\nabla^{k+1}, u_1] \cdot \nabla n_1 \nabla^{k+1} n_1 + ([\nabla^{k+1}, u_1], \nabla u_1) \cdot \nabla^{k+1} u_1 \\
&\quad - \int u_1 \cdot \nabla \nabla^{k+1} n_1 \nabla^{k+1} n_1 + (u_1 \cdot \nabla \nabla^{k+1} u_1) \cdot \nabla^{k+1} u_1 \\
&\leq C(\|\nabla u_1\|_{L^\infty} \|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^{k+1} u_1\|_{L^2} \|\nabla n_1\|_{L^\infty}) \|\nabla^{k+1} n_1\|_{L^2} \\
&\quad + \|\nabla u_1\|_{L^\infty} \|\nabla^{k+1} u_1\|_{L^2} - \frac{1}{2} \int u_1 \cdot \nabla (\nabla^{k+1} n_1 \nabla^{k+1} n_1 + \nabla^{k+1} u_1 \cdot \nabla^{k+1} u_1) \\
&\leq C \|\nabla(n_1, u_1)\|_{L^\infty} \|\nabla^{k+1}(n_1, u_1)\|_{L^2}^2 + \frac{1}{2} \operatorname{div} u_1 \nabla^{k+1} n_1 \nabla^{k+1} n_1 + \operatorname{div} u_1 \nabla^{k+1} u_1 \cdot \nabla^{k+1} u_1 \\
&\leq C \delta (\|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2).
\end{aligned} \tag{2.33}$$

In the same way, one can deduce that

$$J_2 \leq C \delta (\|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2). \tag{2.34}$$

Thus we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int |\nabla^{k+1}(n_1, u_1)|^2 + \|\nabla^{k+1} u_1\|_{L^2}^2 - \int \nabla^{k+1} u_1 \nabla^{k+1} \nabla \phi \\
&\leq C \delta (\|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2).
\end{aligned} \tag{2.35}$$

The similar estimate of n_2, u_2 is

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int |\nabla^{k+1}(n_2, u_2)|^2 + \|\nabla^{k+1} u_2\|_{L^2}^2 + \int \nabla^{k+1} u_2 \nabla^{k+1} \nabla \phi \\
&\leq C \delta (\|\nabla^{k+1} n_2\|_{L^2}^2 + \|\nabla^{k+1} u_2\|_{L^2}^2).
\end{aligned} \tag{2.36}$$

Finally, we give the estimates of the last terms in the left hand side of (2.35) and (2.36) as follows.

$$\begin{aligned}
&- \int \nabla^{k+1} \nabla \phi \cdot \nabla^{k+1} u_1 + \int \nabla^{k+1} \nabla \phi \cdot \nabla^{k+1} u_2 \\
&= \int \nabla^{k+1} (\operatorname{div} u_1) \nabla^{k+1} \phi - \int \nabla^{k+1} (\operatorname{div} u_2) \nabla^{k+1} \phi \\
&= - \int \nabla^{k+1} [\partial_t n_1 + \operatorname{div}(n_1 u_1)] \nabla^{k+1} \phi + \int \nabla^{k+1} [\partial_t n_2 + \operatorname{div}(n_2 u_2)] \nabla^{k+1} \phi \\
&= - \int \nabla^{k+1} \partial_t (n_1 - n_2) \nabla^{k+1} \phi - \int \nabla^{k+1} (\operatorname{div}(n_1 u_1)) \nabla^{k+1} \phi + \int \nabla^{k+1} (\operatorname{div}(n_2 u_2)) \nabla^{k+1} \phi \\
&= - \int \nabla^{k+1} \partial_t \Delta \phi \nabla^{k+1} \phi - \int \nabla^{k+1} (\operatorname{div}(n_1 u_1)) \nabla^{k+1} \phi + \int \nabla^{k+1} (\operatorname{div}(n_2 u_2)) \nabla^{k+1} \phi \\
&= \frac{1}{2} \frac{d}{dt} \|\nabla^{k+1} \nabla \phi\|_{L^2}^2 + \int \nabla^{k+1} (n_1 u_1) \nabla^{k+1} \nabla \phi - \int \nabla^{k+1} (n_2 u_2) \nabla^{k+1} \nabla \phi \\
&:= \frac{1}{2} \frac{d}{dt} \|\nabla^{k+1} \nabla \phi\|_{L^2}^2 + J_3 + J_4.
\end{aligned} \tag{2.37}$$

Using Hölder's inequality and Lemma 2.2 and Cauchy's inequality, we obtain

$$\begin{aligned}
J_3 &= \int \nabla^{k+1}(n_1 u_1) \cdot \nabla^{k+1} \nabla \phi \leq C \|\nabla^{k+1} \nabla \phi\|_{L^2} \|\nabla^{k+1}(n_1 u_1)\|_{L^2} \\
&\leq C \|\nabla^{k+1} \nabla \phi\|_{L^2} (\|n_1\|_{L^\infty} \|\nabla^{k+1} u_1\|_{L^2} + \|u_1\|_{L^\infty} \|\nabla^{k+1} n_1\|_{L^2}) \\
&\leq C \delta (\|\nabla^{k+1} u_1\|_{L^2}^2 + \|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^{k+1} \nabla \phi\|_{L^2}^2).
\end{aligned} \tag{2.38}$$

Similarly, we have

$$\begin{aligned}
J_4 &= \int \nabla^{k+1}(n_2 u_2) \cdot \nabla^{k+1} \nabla \phi \leq C \|\nabla^{k+1} \nabla \phi\|_{L^2} \|\nabla^{k+1}(n_2 u_2)\|_{L^2} \\
&\leq C \delta (\|\nabla^{k+1} u_2\|_{L^2}^2 + \|\nabla^{k+1} n_2\|_{L^2}^2 + \|\nabla^{k+1} \nabla \phi\|_{L^2}^2).
\end{aligned} \tag{2.39}$$

Hence, plugging (2.33), (2.34), (2.37), (2.38) and (2.39) into (2.32), we deduce that (2.31). This proves Lemma 2.11. \blacksquare

Now, we shall recover the dissipation estimate for n_1, n_2 .

Lemma 2.12. Assume that $0 \leq k \leq 2$, then we have

$$\begin{aligned}
&\frac{d}{dt} \left\{ \int \nabla^k u_1 \cdot \nabla \nabla^k n_1 + \nabla^k u_2 \cdot \nabla \nabla^k n_2 \right\} + C \|\nabla^{k+1}(n_1, n_2, \nabla \phi)\|_{L^2}^2 \\
&\leq C (\|\nabla^k u_1\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2 + \|\nabla^k u_2\|_{L^2}^2 + \|\nabla^{k+1} u_2\|_{L^2}^2).
\end{aligned} \tag{2.40}$$

Proof. Let $0 \leq k \leq 2$. Applying ∇^k to (2.1)₂ and then multiplying the resulting equality by $\nabla \nabla^k n_1$, we have

$$\begin{aligned}
\|\nabla^{k+1} n_1\|_{L^2}^2 - \int \nabla \nabla^k n_1 \nabla^k \nabla \phi &\leq - \int \nabla^k \partial_t u_1 \cdot \nabla \nabla^k n_1 + C \|\nabla^k u_1\|_{L^2} \|\nabla^{k+1} n_1\|_{L^2} \\
&\quad + \|\nabla^k(u_1 \cdot \nabla u_1 + h(n_1) \nabla n_1)\|_{L^2} \|\nabla^{k+1} n_1\|_{L^2}.
\end{aligned} \tag{2.41}$$

First, we estimate the first term in the right hand side of (2.39).

$$\begin{aligned}
&- \int \nabla^k u_1 \partial_t u_1 \cdot \nabla \nabla^k n_1 \\
&= - \frac{d}{dt} \int \nabla^k \cdot \nabla \nabla^k n_1 - \int \nabla^k \operatorname{div} u_1 \cdot \nabla^k \partial_t n_1 \\
&= - \frac{d}{dt} \int \nabla^k u_1 \cdot \nabla \nabla^k n_1 + \|\nabla^k \operatorname{div} u_1\|_{L^2}^2 + \int \nabla^k u_1 \operatorname{div} \cdot \nabla^k (u_1 \cdot \nabla n_1 + n_1 \operatorname{div} u_1),
\end{aligned} \tag{2.42}$$

Next, we shall estimate the last two terms in (2.40) by

$$\begin{aligned}
\int \nabla^k \operatorname{div} u_1 \cdot \nabla^k (u_1 \cdot \nabla n_1) &= \int \sum_{0 \leq l \leq k} C_k^l \nabla^l u_1 \cdot \nabla \nabla^{k-l} n_1 \cdot \nabla^k \operatorname{div} u_1 \\
&\leq C \sum_{0 \leq l \leq k} \|\nabla^l u_1 \cdot \nabla \nabla^{k-l} n_1\|_{L^2} \|\nabla^{k+1} u_1\|_{L^2}.
\end{aligned} \tag{2.43}$$

If $l = 0$, then

$$\begin{aligned}
\|u_1 \cdot \nabla \nabla^k n_1\|_{L^2} \|\nabla^{k+1} u_1\|_{L^2} &\leq C \|u_1\|_{L^\infty} \|\nabla^{k+1} n_1\|_{L^2} \|\nabla^{k+1} u_1\|_{L^2} \\
&\leq C \delta (\|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2).
\end{aligned} \tag{2.44}$$

If $1 \leq l \leq [k/2]$, using Hölder's inequality and Lemma 2.1, we get

$$\begin{aligned} \|\nabla^l u_1 \cdot \nabla \nabla^{k-l} n_1\|_{L^2} &\leq C \|\nabla^{k+1-l}\|_{L^6} \|\nabla^l u_1\|_{L^3} \\ &\leq C \|n_1\|_{L^2}^{\frac{l-1}{k+1}} \|\nabla^{k+1} n_1\|_{L^2}^{\frac{k-l+2}{k+1}} \|\nabla^\alpha u_1\|_{L^2}^{\frac{k-l+2}{k+1}} \|\nabla^{k+1} u_1\|_{L^2}^{\frac{l-1}{k+1}} \\ &\leq C \delta (\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^{k+1} u_1\|_{L^2}), \end{aligned} \quad (2.45)$$

where $\alpha = \frac{3k+3}{2k-2l+4} \in [3/2, 3)$, since $l \leq k/2$.

If $[k/2] + 1 \leq l \leq k$, using Hölder's inequality and Lemma 2.1 again, we obtain

$$\begin{aligned} \|\nabla^l u_1 \cdot \nabla \nabla^{k-l} n_1\|_{L^2} &\leq C \|\nabla^{k+1-l}\|_{L^3} \|\nabla^l u_1\|_{L^6} \\ &\leq C \|\nabla^\alpha n_1\|_{L^2}^{\frac{l+1}{k+1}} \|\nabla^{k+1} n_1\|_{L^2}^{\frac{k-l}{k+1}} \|u_1\|_{L^2}^{\frac{k-l}{k+1}} \|\nabla^{k+1} u_1\|_{L^2}^{\frac{l+1}{k+1}} \\ &\leq C \delta (\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^{k+1} u_1\|_{L^2}), \end{aligned} \quad (2.46)$$

where $\alpha = \frac{3k+3}{2l+2} \in [3/2, 3)$, since $l \geq \frac{k+1}{2}$.

Thus, from (2.44), (2.45) and (2.46), we obtain

$$\int \nabla^k \operatorname{div} u_1 \cdot \nabla^k (u_1 \cdot \nabla n_1) \leq C \delta (\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^{k+1} u_1\|_{L^2}). \quad (2.47)$$

Similarly, we also get

$$\int \nabla^k \operatorname{div} u_1 \cdot \nabla^k (n_1 \operatorname{div} u_1) \leq C \delta (\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^{k+1} u_1\|_{L^2}), \quad (2.48)$$

and

$$\|\nabla^k (u_1 \cdot \nabla u_1 + h(n_1) \nabla n_1)\|_{L^2} \leq C \delta (\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^{k+1} u_1\|_{L^2}). \quad (2.49)$$

Hence, by (2.40)-(2.49), we have

$$\begin{aligned} &\frac{d}{dt} \int \nabla^k u_1 \cdot \nabla \nabla^k n_1 + C \|\nabla^{k+1} n_1\|_{L^2}^2 - \int \nabla \nabla^k n_1 \nabla^k \nabla \phi \\ &\leq C (\|\nabla^k u_1\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2). \end{aligned} \quad (2.50)$$

On the other hand, by a method similar to the above, we have

$$\begin{aligned} &\frac{d}{dt} \int \nabla^k u_2 \cdot \nabla \nabla^k n_2 + C \|\nabla^{k+1} n_2\|_{L^2}^2 + \int \nabla \nabla^k n_2 \nabla^k \nabla \phi \\ &\leq C (\|\nabla^k u_2\|_{L^2}^2 + \|\nabla^{k+1} u_2\|_{L^2}^2). \end{aligned} \quad (2.51)$$

Finally, using the Poisson equation in (2.1), the second terms on the left hand side of (2.50) and (2.51) can be estimated as

$$- \int \nabla \nabla^k n_1 \nabla^k \nabla \phi + \int \nabla \nabla^k n_2 \nabla^k \nabla \phi = \frac{1}{2} \|\nabla^{k+1} \nabla \phi\|_{L^2}^2. \quad (2.52)$$

Summing (2.50) and (2.51), and using (2.52), one has

$$\begin{aligned} &\frac{d}{dt} \left\{ \int \nabla^k u_2 \cdot \nabla \nabla^k n_2 + \nabla^k u_1 \cdot \nabla \nabla^k n_1 \right\} + C \|\nabla^{k+1} (n_1, n_2, \nabla \phi)\|_{L^2}^2 \\ &\leq C (\|\nabla^k u_1\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2 + \|\nabla^k u_2\|_{L^2}^2 + \|\nabla^{k+1} u_2\|_{L^2}^2). \end{aligned} \quad (2.53)$$

This proves (2.40). ■

2.3. Estimates in $\dot{H}^{-s}(\mathbb{R}^3)$

The following lemma plays a key role in the proof of Theorem 1.2. It shows an energy estimate of the solutions in the negative Sobolev space $\dot{H}^{-s}(\mathbb{R}^3)$. Namely, we have

Lemma 2.13. If $\|n_{i0}, u_{i0}, \nabla \phi_0\|_{H^3} \ll 1$ with $i = 1, 2$, for $s \in (0, \frac{1}{2}]$, we have

$$\frac{d}{dt} \|(n_i, u_i, \nabla \phi)\|_{\dot{H}^{-s}} \leq C(\|\nabla n_i\|_{H^1}^2 + \|u_i\|_{H^2}^2) \|(n_i, u_i, \nabla \phi)\|_{\dot{H}^{-s}}, \quad i = 1, 2; \quad (2.54)$$

and for $s \in (\frac{1}{2}, \frac{3}{2})$, we have

$$\begin{aligned} & \frac{d}{dt} \|(n_i, u_i, \nabla \phi)\|_{\dot{H}^{-s}} \\ & \leq C \left\{ \|(n_i, u_i)\|_{L^2}^{s-\frac{1}{2}} \|\nabla(n_i, u_i)\|_{H^1}^{\frac{5}{2}-s} + \|u_i\|_{L^2} \|n_i\|_{L^2}^{s-\frac{1}{2}} \|\nabla n_i\|_{L^2}^{\frac{3}{2}-s} \right\} \|(n_i, u_i, \nabla \phi)\|_{\dot{H}^{-s}}, \quad i = 1, 2. \end{aligned} \quad (2.55)$$

Proof. Applying Λ^{-s} to (2.2)₁, (2.2)₂ and multiplying the resulting identity by $\Lambda^{-s}n_1$, $\Lambda^{-s}u_1$, respectively, and integrating over \mathbb{R}^3 by parts, we get

$$\begin{aligned} & \frac{d}{dt} \int (|\Lambda^{-s}n_i|^2 + |\Lambda^{-s}u_i|^2) + \int |\nabla \Lambda^{-s}u_i|^2 + (-1)^i \int \Lambda^{-s} \nabla \phi \cdot \Lambda^{-s}u_i \\ & = \int \Lambda^{-s}(-n_i \operatorname{div} u_i - u_i \cdot \nabla n_i) \Lambda^{-s}n_i - \Lambda^{-s}(u_i \cdot \nabla u_i + h(n_i) \nabla n_i) \cdot \Lambda^{-s}u_i \\ & \leq C \|n_i \operatorname{div} u_i + u_i \cdot \nabla n_i\|_{\dot{H}^{-s}} \|n_i\|_{\dot{H}^{-s}} + \|u_i \cdot \nabla u_i + h(n_i) \nabla n_i\|_{\dot{H}^{-s}} \|u_i\|_{\dot{H}^{-s}}, \end{aligned} \quad (2.56)$$

If $s \in (0, 1/2]$, then by Lemma 2.1, Lemma 2.3 and Young's inequality, the right hand side of (2.56) can be estimated as follows.

$$\begin{aligned} \|n_i \operatorname{div} u_i\|_{\dot{H}^{-s}} & \leq C \|n_i \operatorname{div} u_i\|_{L^{\frac{1}{1/2+s/3}}} \leq C \|n_i\|_{L^{3/s}} \|\nabla u_i\|_{L^2} \\ & \leq C \|\nabla n_i\|_{L^2}^{1/2+s} \|\nabla^2 n_i\|_{L^2}^{1/2-s} \|\nabla u_i\|_{L^2} \\ & \leq C(\|\nabla n_i\|_{H^1}^2 + \|\nabla u_i\|_{L^2}^2), \end{aligned} \quad (2.57)$$

where we have used the facts $\frac{1}{2} + \frac{s}{3} < 1$ and $\frac{3}{s} \geq 6$.

Similarly, it holds that

$$\|u_i \cdot \nabla n_i\|_{\dot{H}^{-s}} \leq C(\|\nabla u_i\|_{H^1}^2 + \|\nabla n_i\|_{L^2}^2); \quad (2.58)$$

$$\|u_i \cdot \nabla u_i\|_{\dot{H}^{-s}} \leq C(\|\nabla u_i\|_{H^1}^2 + \|\nabla u_i\|_{L^2}^2); \quad (2.59)$$

$$\|h(n_i) \cdot \nabla n_i\|_{\dot{H}^{-s}} \leq C(\|\nabla n_i\|_{H^1}^2 + \|\nabla n_i\|_{L^2}^2). \quad (2.60)$$

Now if $s \in (1/2, 3/2)$, then $1/2 + s/3 < 1$ and $2 < 3/s < 6$. We shall estimate the right hand side of (2.55) in a different way. Using Sobolev's inequality, we have

$$\begin{aligned} \|n_i \operatorname{div} u_i\|_{\dot{H}^{-s}} & \leq C \|n_i \operatorname{div} u_i\|_{L^{\frac{1}{1/2+s/3}}} \leq C \|n_i\|_{L^{3/s}} \|\nabla u_i\|_{L^2} \\ & \leq C \|n_i\|_{L^2}^{s-1/2} \|\nabla n_i\|_{L^2}^{3/2-s} \|\nabla u_i\|_{L^2}, \end{aligned} \quad (2.61)$$

where we have used the facts $\frac{1}{2} + \frac{s}{3} < 1$ and $\frac{3}{s} \geq 6$.

Similarly, it holds for $s \in (1/2, 3/2)$ that

$$\|u_i \cdot \nabla n_i\|_{\dot{H}^{-s}} \leq C \|u_i\|_{L^2}^{s-1/2} \|\nabla u_i\|_{L^2}^{3/2-s} \|\nabla n_i\|_{L^2}; \quad (2.62)$$

$$\|u_i \cdot \nabla u_i\|_{\dot{H}^{-s}} \leq C \|u_i\|_{L^2}^{s-1/2} \|\nabla u_i\|_{L^2}^{3/2-s} \|\nabla u_i\|_{L^2}; \quad (2.63)$$

$$\|h(n_i) \cdot \nabla n_i\|_{\dot{H}^{-s}} \leq C \|n_i\|_{L^2}^{s-1/2} \|\nabla n_i\|_{L^2}^{3/2-s} \|\nabla n_i\|_{L^2}. \quad (2.64)$$

Finally, we turn to the last term in the left hand side of (2.56) with $i = 1, 2$. We have

$$\begin{aligned} & - \int \Lambda^{-s} \nabla \phi \cdot \Lambda^{-s} u_1 + \int \Lambda^{-s} \nabla \phi \cdot \Lambda^{-s} u_2 \\ &= \int \Lambda^{-s} \phi \Lambda^{-s} \operatorname{div} u_1 - \int \Lambda^{-s} \phi \Lambda^{-s} \operatorname{div} u_2 \\ &= - \int \Lambda^{-s} \phi \Lambda^{-s} \partial_t (n_1 - n_2) + \int \Lambda^{-s} \phi \Lambda^{-s} \operatorname{div} (n_1 u_1 - n_2 u_2) \\ &= \frac{1}{2} \frac{d}{dt} \int |\Lambda^{-s} \nabla \phi|^2 - \int \Lambda^{-s} \nabla \phi \cdot \Lambda^{-s} (n_1 u_1 - n_2 u_2). \end{aligned} \quad (2.65)$$

If $s \in (0, 1/2)$, we use Lemma 2.1 and Lemma 2.4 to obtain

$$\begin{aligned} \|\Lambda^{-s} (n_i u_i)\|_{L^2} &\leq C \|u_i\|_{L^2} \|n_i\|_{L^{3/s}} \leq C \|u_i\|_{L^2} \|\nabla n_i\|_{L^2}^{1/2-s} \|\nabla^2 n_i\|_{L^2}^{1/2+s} \\ &\leq C (\|u_i\|_{L^2}^2 + \|\nabla n_i\|_{H^1}^2); \end{aligned} \quad (2.66)$$

and if $s \in (1/2, 3/2)$, we have

$$\|\Lambda^{-s} (n_i u_i)\|_{L^2} \leq C \|u_i\|_{L^2} \|n_i\|_{L^{3/s}} \leq C \|u_i\|_{L^2} \|\nabla n_i\|_{L^2}^{s-1/2} \|\nabla^2 n_i\|_{L^2}^{3/2-s}. \quad (2.67)$$

Consequently, in light of (2.56)-(2.67), and using Young's inequality, we deduce (2.54) and (2.55). \blacksquare

2.4. Estimates in $\dot{B}_{2,\infty}^{-s}(\mathbb{R}^3)$

In this subsection, we will derive the evolution of the negative Besov norms of the solutions. The argument is similar to the previous subsection.

Lemma 2.14. If $\|n_{i0}, u_{i0}, \nabla \phi_0\|_{H^3} \ll 1$ with $i = 1, 2$, for $s \in (0, \frac{1}{2}]$, we have

$$\frac{d}{dt} \|(n_i, u_i, \nabla \phi)\|_{\dot{B}_{2,\infty}^{-s}}^2 \leq C (\|\nabla n_i\|_{H^1}^2 + \|u_i\|_{H^2}^2) \|(n_i, u_i, \nabla \phi)\|_{\dot{B}_{2,\infty}^{-s}}, \quad i = 1, 2; \quad (2.68)$$

and for $s \in (\frac{1}{2}, \frac{3}{2}]$, we have

$$\begin{aligned} & \frac{d}{dt} \|(n_i, u_i, \nabla \phi)\|_{\dot{B}_{2,\infty}^{-s}}^2 \\ & \leq C \left\{ \|(n_i, u_i)\|_{L^2}^{s-\frac{1}{2}} \|\nabla(n_i, u_i)\|_{H^1}^{\frac{5}{2}-s} + \|u_i\|_{L^2} \|n_i\|_{L^2}^{s-\frac{1}{2}} \|\nabla n_i\|_{L^2}^{\frac{3}{2}-s} \right\} \|(n_i, u_i, \nabla \phi)\|_{\dot{B}_{2,\infty}^{-s}}, \quad i = 1, 2. \end{aligned} \quad (2.69)$$

Proof. Applying $\dot{\Delta}_j$ to (2.2)₁, (2.2)₂ and multiplying the resulting identity by $\dot{\Delta}_j n_1$, $\dot{\Delta}_j u_1$, respectively, and integrating over \mathbb{R}^3 by parts, we get

$$\begin{aligned} & \frac{d}{dt} \int (|\dot{\Delta}_j n_1|^2 + |\dot{\Delta}_j u_1|^2) + \int |\nabla \dot{\Delta}_j u|^2 - \int \dot{\Delta}_j \nabla \phi \cdot \dot{\Delta}_j u_1 \\ &= \int \dot{\Delta}_j (-n_1 \operatorname{div} u_1 - u_1 \cdot \nabla n_1) \dot{\Delta}_j n_1 - \dot{\Delta}_j (u_1 \cdot \nabla u_1 + h(n_1) \nabla n_1) \cdot \dot{\Delta}_j u_1 \\ &\leq C \|n_1 \operatorname{div} u_1 + u_1 \cdot \nabla n_1\|_{\dot{B}_{2,\infty}^{-s}} \|n_1\|_{\dot{B}_{2,\infty}^{-s}} + \|u_1 \cdot \nabla u_1 + h(n_1) \nabla n_1\|_{\dot{B}_{2,\infty}^{-s}} \|u_1\|_{\dot{B}_{2,\infty}^{-s}}. \end{aligned} \quad (2.70)$$

Then, as the proof of Lemma 2.13, applying Lemma 2.6 instead to estimate the $\dot{B}_{2,\infty}^{-s}$ norm, we complete the proof of Lemma 2.14. \blacksquare

3. Proof of Theorems

3.1. Proof of Theorem 1.1

In this subsection, we shall use the energy estimates in Subsection 2.2 to prove the global existence in H^3 norm.

We first close the energy estimates at each l -th level to prove (1.3). Let $0 \leq l \leq 2$. Summing up the estimates (2.4) from $k = l$ to $k = 2$, and then adding the resulting estimates to (2.31) for $k = 2$, by changing the index and since $\delta \ll 1$, we have

$$\begin{aligned} & \frac{d}{dt} \sum_{l \leq k \leq 3} \|\nabla^k(n_1, u_1, n_2, u_2, \nabla \phi)\|_{L^2}^2 + C_1 \sum_{l \leq k \leq 3} \|\nabla^k(u_1, u_2)\|_{L^2}^2 \\ & \leq C_2 \delta \sum_{l+1 \leq k \leq 3} \|\nabla^k(n_1, n_2, \nabla \phi)\|_{L^2}^2. \end{aligned} \quad (3.1)$$

Summing up (2.40) of Lemma 2.12 from $k = l$ to 2, we have

$$\begin{aligned} & \frac{d}{dt} \sum_{l \leq k \leq 2} \int (\nabla^k u_1 \cdot \nabla \nabla^k n_1 + \nabla^k u_2 \cdot \nabla \nabla^k n_2) + C_3 \sum_{l+1 \leq k \leq 3} \|\nabla^k(n_1, n_2, \nabla \phi)\|_{L^2}^2 \\ & \leq C_4 \sum_{l \leq k \leq 3} \|\nabla^k(u_1, u_2)\|_{L^2}^2. \end{aligned} \quad (3.2)$$

Making a calculus $2C_2\delta/C_3 \times (3.2) + (3.1)$, and by using the fact $\delta \ll 1$, we can conclude that there exists a constant $C_5 > 0$ such that for $0 \leq l \leq 2$,

$$\begin{aligned} & \frac{d}{dt} \left\{ \sum_{l \leq k \leq 3} \|\nabla^k(n_1, u_1, n_2, u_2, \nabla \phi)\|_{L^2}^2 + \frac{2C_2\delta}{C_3} \sum_{l \leq k \leq 2} \int (\nabla^k u_1 \cdot \nabla \nabla^k n_1 + \nabla^k u_2 \cdot \nabla \nabla^k n_2) \right\} \\ & + C_5 \left\{ \sum_{l \leq k \leq 3} \|\nabla^k(u_1, u_2)\|_{L^2}^2 + \sum_{l+1 \leq k \leq 3} \|\nabla^k(n_1, n_2, \nabla \phi)\|_{L^2}^2 \right\} \leq 0. \end{aligned} \quad (3.3)$$

by the smallness of δ and using Cauchy's inequality, we deduce that

$$\begin{aligned} & C_6^{-1} \|\nabla^l(n_1, u_1, n_2, u_2, \nabla \phi)\|_{H^{3-l}}^2 \\ & \leq \sum_{l \leq k \leq 3} \|\nabla^k(n_1, u_1, n_2, u_2, \nabla \phi)\|_{L^2}^2 + \frac{2C_2\delta}{C_3} \sum_{l \leq k \leq 2} \int (\nabla^k u_1 \cdot \nabla \nabla^k n_1 + \nabla^k u_2 \cdot \nabla \nabla^k n_2) \\ & \leq C_6 \|\nabla^l(n_1, u_1, n_2, u_2, \nabla \phi)\|_{H^{3-l}}^2, \quad 0 \leq l \leq 2. \end{aligned} \quad (3.4)$$

As a result, we have the following estimate in Sobolev's space for $0 \leq l \leq 2$

$$\frac{d}{dt} \|\nabla^l(n_1, u_1, n_2, u_2, \nabla \phi)\|_{H^{3-l}}^2 + \left\{ \|\nabla^l(u_1, u_2)\|_{H^{3-l}}^2 + \|\nabla^{l+1}(n_1, n_2, \nabla \phi)\|_{H^{2-l}}^2 \right\} \leq 0. \quad (3.5)$$

Taking $l = 0$ in (3.5), and integrating directly in time, we have

$$\|(n_1, u_1, n_2, u_2, \nabla \phi)\|_{H^3}^2 \leq C_6^2 \|(n_{10}, u_{10}, n_{20}, u_{20}, \nabla \phi_0)\|_{H^3}^2. \quad (3.6)$$

By a standard continuity argument, since $\|(n_{10}, u_{10}, n_{20}, u_{20}, \nabla \phi_0)\|_{H^3}$ is sufficiently small, this closes the a priori estimates (2.2). Thus we obtain the global existence in Theorem 1.1.

3.2. Proof of Theorem 1.2

In this subsection, we will prove the optimal time decay rates of the unique global solution to system (2.1) in theorem 1.2.

First, from Lemma 2.13, we need distinct the arguments by the value of s . For $s \in [0, 1/2)$, integrating (2.53) in time, by (1.3), we have

$$\begin{aligned} \|(n_i, u_i, \nabla \phi)\|_{\dot{H}^{-s}}^2 &\leq \|(n_{i0}, u_{i0}, \nabla \phi_0)\|_{\dot{H}^{-s}}^2 + C \int_0^t \|\nabla(n_i, u_i)\|_{H^1}^2 \|(n_i, u_i, \nabla \phi)\|_{\dot{H}^{-s}} d\tau \\ &\leq C_0(1 + \sup_{0 \leq \tau \leq t} \{ \|(n_i, u_i, \nabla \phi)\|_{\dot{H}^{-s}} \}). \end{aligned} \quad (3.7)$$

This yields

$$\|(n_1, u_1, n_2, u_2, \nabla \phi)\|_{\dot{H}^{-s}} \leq C_0 \text{ for } s \in [0, 1/2]. \quad (3.8)$$

Using Lemma 2.14, we similarly have

$$\|(n_1, u_1, n_2, u_2, \nabla \phi)\|_{\dot{B}_{2,\infty}^{-s}} \leq C_0 \text{ for } s \in (0, 1/2]. \quad (3.9)$$

If $0 \leq l \leq 2$, we may use Lemma 2.3 to have

$$\|\nabla^{l+1} f\|_{L^2} \geq C \|f\|_{\dot{H}^s}^{-\frac{1}{l+s}} \|\nabla^l f\|_{L^2}^{1+\frac{1}{l+s}}. \quad (3.10)$$

By this fact and (3.9), we find

$$\|\nabla^{l+1}(n_1, n_2, \nabla \phi)\|_{L^2}^2 \geq C_0 (\|\nabla^l(n_1, n_2, \nabla \phi)\|_{L^2}^2)^{1+\frac{1}{l+s}}. \quad (3.11)$$

This together with (1.3) yields for $l = 0, 1, 2$,

$$\|\nabla^l(u_1, u_2), \nabla^{l+1}(n_1, n_2, \nabla \phi)\|_{H^{3-l}}^2 \geq C_0 (\|\nabla^l(u_1, u_2, n_1, n_2, \nabla \phi)\|_{H^{3-l}}^2)^{1+\frac{1}{l+s}}. \quad (3.12)$$

Hence, from (3.5), we have the following time differential inequality for $l = 0, 1, 2$

$$\frac{d}{dt} \|\nabla^l(u_1, u_2, n_1, n_2, \nabla \phi)\|_{H^{3-l}}^2 + C_0 (\|\nabla^l(u_1, u_2, n_1, n_2, \nabla \phi)\|_{H^{3-l}}^2)^{1+\frac{1}{l+s}} \leq 0, \quad (3.13)$$

which gives

$$\|\nabla^l(u_1, u_2, n_1, n_2, \nabla \phi)\|_{H^{3-l}}^2 \leq C_0(1+t)^{-(l+s)}, \quad l = 0, 1, 2; \quad s \in [0, \frac{1}{2}]. \quad (3.14)$$

For $s \in (1/2, 3/2)$. Notice that the arguments for the case $s \in [0, 1/2]$ can not be applied to this case (See Lemma 2.13). Observing that we have $n_{10}, u_{10}, n_{20}, u_{20}, \nabla \phi_0 \in \dot{H}^{-1/2}$ since $\dot{H}^{-s} \cap L^2 \subset \dot{H}^{-s'}$ for any $s' \in [0, s]$, we then deduce from what we have proved for (1.6) with $s = 1/2$ that the following decay result holds:

$$\|\nabla^l(n_1, u_1, n_2, u_2, \nabla \phi)\|_{H^{3-l}} \leq C_0(1+t)^{-\frac{l+\frac{1}{2}}{2}} \text{ for } l = 0, 1, 2. \quad (3.15)$$

Integrating (2.53) in time, for $s \in (1/2, 3/2)$, we have

$$\begin{aligned} \|(n_i, u_i, \nabla \phi)\|_{\dot{H}^{-s}} &\leq \|(n_{i0}, u_{i0}, u_{20}, \nabla \phi_0)\|_{\dot{H}^{-s}} \\ &\quad + C \int_0^t \left\{ \|(n_i, u_i)\|_{L^2}^{s-\frac{1}{2}} \|\nabla(n_i, u_i)\|_{H^1}^{\frac{5}{2}-s} + \|u_i\|_{L^2} \|n_i\|_{L^2}^{s-\frac{1}{2}} \|\nabla n_i\|_{L^2}^{\frac{3}{2}-s} \right\} \|(n_i, u_i, \nabla \phi)\|_{\dot{H}^{-s}} d\tau \\ &\leq \|(n_{i0}, u_{i0}, u_{20}, \nabla \phi_0)\|_{\dot{H}^{-s}} + C \sup_{0 \leq \tau \leq t} \{ \|(n_i, u_i, \nabla \phi)\|_{\dot{H}^{-s}} \} \\ &\quad \times \int_0^t \left\{ \|(n_i, u_i)\|_{L^2}^{s-\frac{1}{2}} \|\nabla(n_i, u_i)\|_{H^1}^{\frac{5}{2}-s} + \|u_i\|_{L^2} \|n_i\|_{L^2}^{s-\frac{1}{2}} \|\nabla n_i\|_{L^2}^{\frac{3}{2}-s} \right\} d\tau \\ &:= \|(n_{i0}, u_{i0}, u_{20}, \nabla \phi_0)\|_{\dot{H}^{-s}} + C \sup_{0 \leq \tau \leq t} \{ \|(n_i, u_i, \nabla \phi)\|_{\dot{H}^{-s}} \} \cdot (K_1 + K_2). \end{aligned} \quad (3.16)$$

For K_1 , by using (3.15), we deduce that for the case $s \in (\frac{1}{2}, \frac{3}{2})$

$$\begin{aligned} K_1 &= C \int_0^t \{ \| (n_i, u_i) \|_{L^2}^{s-\frac{1}{2}} \| \nabla (n_i, u_i) \|_{H^1}^{\frac{5}{2}-s} \} \| (n_i, u_i, \nabla \phi) \|_{\dot{H}^{-s}} d\tau \\ &\leq C_0 + C_0 \int_0^t (1+\tau)^{-7/4-s/2} d\tau \sup_{0 \leq \tau \leq t} \{ \| (n_i, u_i, \nabla \phi) \|_{\dot{H}^{-s}} \} \\ &\leq C_0 \left\{ 1 + \sup_{0 \leq \tau \leq t} \{ \| (n_i, u_i, \nabla \phi) \|_{\dot{H}^{-s}} \} \right\}, \quad i = 1, 2; \quad s \in (\frac{1}{2}, \frac{3}{2}). \end{aligned} \quad (3.17)$$

For K_2 , we must distinct the arguments by the value of s : $s \in (\frac{1}{2}, 1)$ and $s \in [1, \frac{3}{2})$. When $s \in (\frac{1}{2}, 1)$,

$$\begin{aligned} K_2 &= C \int_0^t \{ \| u_i \|_{L^2} \| n_i \|_{L^2}^{s-\frac{1}{2}} \| \nabla n_i \|_{L^2}^{\frac{3}{2}-s} \} \| (n_i, u_i, \nabla \phi) \|_{\dot{H}^{-s}} d\tau \\ &\leq C \left\{ \int_0^t \| u_i \|_{L^2}^2 d\tau + \int_0^t \| n_i \|_{L^2}^{2s-1} \| \nabla n_i \|_{L^2}^{3-2s} d\tau \right\} \\ &\leq CC_0 + C_0 \int_0^t (1+\tau)^{-\frac{1}{4}(2s-1)} (1+\tau)^{-\frac{3}{4}(3-2s)} d\tau \\ &\leq CC_0 + \int_0^t (1+\tau)^{-\frac{1}{4}(8-4s)} d\tau \leq CC_0, \quad s \in (\frac{1}{2}, 1). \end{aligned} \quad (3.18)$$

Thus, (3.16)-(3.18) imply that

$$\| (n_i, u_i, \nabla \phi) \|_{\dot{H}^{-s}} \leq CC_0, \quad s \in [0, 1). \quad (3.19)$$

Hence (3.19) together with a similar argument as the case $s \in [0, \frac{1}{2}]$, we know the decay result (1.6) is established for any $s \in [0, 1)$:

$$\| \nabla^l (u_1, u_2, n_1, n_2, \nabla \phi) \|_{H^{3-l}}^2 \leq C_0 (1+t)^{-(l+s)}, \quad l = 0, 1, 2; \quad s \in [0, 1). \quad (3.20)$$

Now we choose a constant $s_1 = \frac{5}{8} + \frac{s}{4}$ with $s \in [1, \frac{3}{2})$, then $s_1 < 1$. Then, (3.20) gives

$$\| \nabla^l (u_1, u_2, n_1, n_2, \nabla \phi) \|_{H^{3-l}}^2 \leq C_0 (1+t)^{-(l+s_1)}, \quad l = 0, 1, 2; \quad s_1 \in [0, 1). \quad (3.21)$$

By (3.21), we can prove the decay result for $s \in [1, \frac{3}{2})$. In fact,

$$\begin{aligned} K_2 &= C \int_0^t \{ \| u_i \|_{L^2} \| n_i \|_{L^2}^{s-\frac{1}{2}} \| \nabla n_i \|_{L^2}^{\frac{3}{2}-s} \} \| (n_i, u_i, \nabla \phi) \|_{\dot{H}^{-s}} d\tau \\ &\leq CC_0 \int_0^t (1+\tau)^{-\frac{s_1}{2}} (1+\tau)^{-\frac{s_1}{2}(s-\frac{1}{2})} (1+\tau)^{-\frac{1+s_1}{2}(\frac{3}{2}-s)} d\tau \\ &= CC_0 \int_0^t (1+\tau)^{s_1+\frac{3}{4}-\frac{s}{2}} d\tau = CC_0 \int_0^t (1+\tau)^{\frac{11}{8}-\frac{s}{4}} d\tau \leq CC_0, \quad s \in [1, \frac{3}{2}). \end{aligned} \quad (3.22)$$

Hence, (3.16), (3.17) and (3.22) suffice for that

$$\| (n_i, u_i, \nabla \phi) \|_{\dot{H}^{-s}} \leq CC_0, \quad s \in [0, \frac{3}{2}). \quad (3.23)$$

With (3.23) in hand, we repeat the arguments leading to (1.6) for $s \in [0, 1/2]$ to prove that it hold also for $s \in (1/2, 3/2)$.

Lastly, by using Lemma 2.5, Lemma 2.7, Lemma 2.8, Lemma 2.9 and Lemma 2.14, a similar argument as leading to the estimate (3.23) for the negative Sobolev space can immediately yields that in the negative Besov's space:

$$\| (n_i, u_i, \nabla \phi) \|_{\dot{B}_{2,\infty}^{-s}} \leq CC_0, \quad s \in (0, \frac{3}{2}]. \quad (3.24)$$

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